International Journal of Theoretical Physics, Vol. 33, No. 1, 1994

Representation of Complete Ortholattices by Sets with Orthogonality

Pavel Konôpka[1]

Received July 22, 1993

We consider an algebraic closure operator induced by orthogonality on an arbitrary set and we investigate some problems with regard to the orthomodular law for a complete lattice of closed subsets.

The basic idea of our approach is the arrangement of a representation of a complete lattice by an algebraic closure operator, more precisely by a lattice of closed subsets. There is a modification (Birkhoff, 1967) for complete ortholattices based on a closure operator induced by the orthogonality relation. The formal definition is:

Let V be an arbitrary nonempty set with a relation of orthogonality (we will use designation $\perp$) such that we require symmetricity and irreflexivity (i.e., $x \perp y \Leftrightarrow y \perp x$ and $\forall x \in V$, $x \perp x$ is not true).

A closure operator on the system of all subsets of V is defined $\mathrm{CL}(A) = A^{\perp\perp}$ such that $A^{\perp} = \{z \in V;\ \forall x \in A,\ x \perp z\}$. It is easy to see that CL is closure operator: $A \subseteq \mathrm{CL}(A)$ and $\mathrm{CL}^2(A) = \mathrm{CL}(A)$ and $A \subseteq B \Rightarrow \mathrm{CL}(A) \subseteq \mathrm{CL}(B)$.

The reader can find in Birkhoff (1967) that the system of all closed subsets (the sets which are equal to its closure) forms a complete ortholattice. The problem of representation of an arbitrary complete ortholattice is investigated in McLaren (1964). In our approach we describe another representation such that our representation is "maximal" in some sense. Let us use the language of graph theory.

[1]Department of Mathematics, Faculty of Human and Natural Sciences, M. Bel University, Tajovského 40, 975 49 Banská Bystrica, Slovakia.

0020-7748/94/0100-0013$07.00/0

The complete graph is a graph such that every couple $\{x, y\}$ is in the edge set.

Any orthogonality relation can be represented by graph with vertex set the basic set of orthogonality and x is orthogonal to y is represented by the edge $\{x, y\}$.

The relation of complementarity is defined: (G_1, E_1) and (G_2, E_2) are complementary graphs if $G_1 = G_2$ is the relation between vertex sets and sets of edges E_1 and E_2 form a decomposition of the set of edges of a complete graph with vertex set $G_1 = G_2$. An induced subgraph is a restriction of graph (G, E) to (H, F) such that $H \subseteq G$ and $F = E/H$. The relation between graphs $(G, E) \prec (H, F)$ if there is an isomorphic copy of (G, E) as an induced subgraph of (H, F) is a pseudoorder relation (i.e., antisymmetry is not required). We will use this relation on finite graphs and in case of finite graphs antisymmetry holds if we do not distinguish isomorphic graphs.

We start from an arbitrary complete ortholattice $(L, \leq, ', 0, 1)$. We define the orthogonality relation on $L - \{0, 1\}$:

$$u \perp v \Leftrightarrow u \leq v'$$

It is known (Zapatrin, 1990) that the lattice of closed subsets induced by this orthogonality relation is an isomorphic copy of the starting lattice.

If we start from an arbitrary orthogonality relation to a lattice of closed subsets and apply our procedure of creation of a "new" orthogonality relation on this lattice, we obtain an orthogonality relation such that the starting relation is an induced subgraph of the terminating relation. The embedding map is: $x \to \{x\}^{\perp\perp}$.

This is an exact formulation of the fact that there is a "maximal" orthogonality relation in a system of all orthogonality relations with the same lattice of closed subsets.

There are representations of an arbitrary complete ortholattice such that every element a of lattice is a joint of a set of join-irreducible elements which are smaller than a (McLaren, 1964; Zapatrin, 1990, n.d.). If we assume the existence of this system of join-irreducible elements for every element, we obtain a "minimal" orthogonality relation in the sense of induced subgraphs.

If we try to analyze the system of all orthogonality relations belonging to some complete ortholattice, the question of isolated points of the orthogonality relation can be posed. There is only one closed set in a lattice of closed subsets with the property "an isolated point is element of this set." It is the maximal set of the lattice of closed subsets. This implies that we can remove isolated points without influence on the terminating lattice.

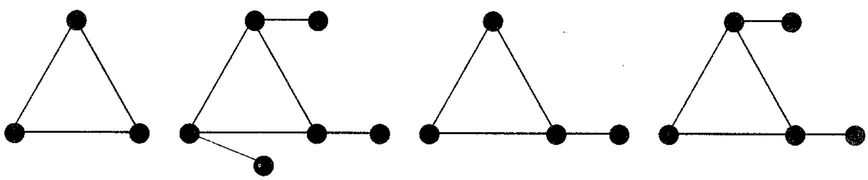
The relation on V [$(V, \perp)$ is the set with orthogonality] defined by

$$x \equiv y \Leftrightarrow \{x\}^{\perp} = \{y\}^{\perp}$$

is a relation of equivalence. This equivalence relation is compatible with orthogonality and there is a natural way to factorize the starting system with orthogonality. We obtain closed subsets in a "new" system with orthogonality from "old" closed subsets. The basic idea is the following: For every closed subset A of V and for every class $[x]$ of equivalence relation the fact that the intersection of these sets is nonempty implies that $[x]$ is subset of A. Closed subsets are unions of equivalence classes. A "new" closed subset of the factor system is a set of classes contained in the "old" closed subset. This mapping is an ortholattice isomorphism. This conclusion allows us to remove redundant points from the graph of the orthogonality relation.

We can pose the universal assumption for this article: We will consider orthogonality relations without isolated points and without redundant points in the sense of the above factorization.

Example 1. We consider a lattice of all subsets of X with card$(X) = 3$. There are four orthogonality relations with this lattice as result. They are represented by diagrams



I II III IV

Diagram I is the minimal relation obtained from joint-irreducible elements. Diagram II is the maximal relation obtained by the construction described above. Diagrams III and IV are diagrams contained in II and containing I. If we analyze the interval of graphs with respect to the order relation induced by induced subgraphs, we obtain that these four relations are all relations in an interval bounded by I and II.

We try to solve the problem to characterize those lattices whose system orthogonality relations are in the interval between the minimal relation obtained by joint-irreducible elements and the maximal relation obtained by the procedure described above. The solution is: A class of all finite ortholattices is a class with this quality.

The proof of this proposition is: If we take an induced subgraph of some graph and we create both lattices of closed subsets, the lattice induced by the subgraph is an isomorphic copy of a certain subposet of the lattice of the starting graph. Isomorphism is considered in the sense of the theory of posets; we do not assume a sublattice or subortholattice. The embedding map is an "identitylike" map. The image of the orthocomplement is greater than or equal to the orthocomplement of the image of the starting element.

If there is an orthogonality system between the minimal and maximal system in the sense of ordering on graphs such that the lattice L_2 is other than the lattice of the maximal and minimal graphs of orthogonality (it is the same lattice L_1), then there are embeddings $\varphi: L_1 \to L_2$ and $\psi: L_2 \to L_1$ and this implies in the case of finite lattices that L_1 and L_2 are isomorphic posets and the not preserving of the orthocomplement implies the existence of a chain of infinite cardinality—contradiction. Lattices L_1 and L_2 must be orthoisomorphic. This completes the proof.

There is the natural question of enlarging the class of finite lattices to a more general class. There is a counterexample in a class of ortholattices with cardinality of all chains bounded by 4. It is a horizontal sum of countably many of algebras of type 2^3 and the horizontal sum of the above lattice with algebra of type 2^2. There is an orthogonality relation of the last lattice in the interval of orthogonalities of first lattice. Enlargement is not too substantial.

This reflection warrants the following procedure: We start from an arbitrary finite graph, we create a lattice of closed subsets (it is finite); we find a system of all joint-irreducible elements of the last lattice and we consider this system with natural orthocomplementation as a representation of this lattice by the graph. The graphic form of the last system with orthogonality is well known, the so-called Greechie diagram.

There is an important algebraic property for quantum logic applications—the orthomodular law. The problem of testing orthomodularity was posed and partly solved by Zapatrin (1990, n.d.). In our approach the problem can be formulated: If we start from the orthogonality system and we use the procedure described above, what is the danger of a "bad" choice of the starting orthogonality system for obtaining an ortholattice which is orthomodular?

In the language of graph theory a nice characterization is considered a characterization by forbidden subgraphs. This nice characterization is not possible in this case. We will use the notion of a clique: A clique is a maximal complete subgraph of a graph.

Proposition 1. If there are two cliques of cardinality 2 with common element, then the induced lattice is not orthomodular.

Proof. We denote these cliques $\{a, b\}$ and $\{b, c\}$. Now $a \in \{b\}^{\perp}$. Similarly $c \in \{b\}^{\perp}$. The orthomodularity law implies $\{a\}^{\perp\perp} = \{b\}^{\perp} = \{c\}^{\perp\perp}$. We compute

$$\{b\}^{\perp} = (\{a\}^{\perp\perp} \cup [\{b\}^{\perp} \cap \{a\}^{\perp}])^{\perp\perp}$$

$\{a, b\}$ is a clique, which implies $\{b\}^{\perp} \cap \{a\}^{\perp} = \varnothing$ and we use the property of closure to finish the proof. The conclusion is in contradiction with the universal assumption ($\equiv$)—factorization.

This criterion is very understandable, but it is not too effective. The next criterion may be induced from the notion of the interior of a subset of a set with orthogonality. The definition of the interior is

$$\mathrm{Int}(A) = (A)^{c\perp\perp c}$$

where $(A)^{c}$ is the set-theoretic complement of A in V.

We omit detailed reflection about the interior. This operator allows us to introduce a lattice of open subsets (i.e., it is equal to its interior). A lattice of open subsets is isomorphic to a lattice of closed subsets and both form the dual couple of subposets of the power set of the basic set V. Every set such that $A^{\perp} = A^{c}$ is an element of intersection of systems of closed and open subsets. The subsets with the last property we call strongly regular. The subsets from the intersection of systems of closed and open subsets we call regular (interior is equal to closure).

Proposition 2. If a system with orthogonality contains a regular subset which is not strongly regular, then the lattice of closed subsets is not orthomodular.

Proof. Let A be a subset which is regular but not strongly regular. This implies $A^{c\perp\perp} \cap A^{\perp} = \varnothing$. From the orthomodularity law we compute $A^{c\perp\perp} = A^{\perp}$, which is in contradiction with the assumption that A is not strongly regular. We conclude the falsity of orthomodularity law.

This criterion is stronger than the first criterion, but there are important cases in which it is not effective.

Example 2. Let $(V, \perp)$ be a system with orthogonality induced by the Greechie diagram of 4-loop of algebras of type 2^3. By a lemma of Greechie, the orthomodular poset with this diagram is not a lattice. A lattice of closed subsets is an ortholattice, but it is not orthomodular. Strongly regular sets are only $\varnothing$ and V, and regular sets similarly. Cliques of cardinality 2 do not exist. Both criteria are not effective.

We obtain a partial solution of the orthomodularity problem by the assumption of the existence of finite common delimitation of the cardinality of cliques. Our hypothesis is that there exist forbidden subgraphs for orthomodularity in a class of bounded cardinality of cliques.

By routine computing we can verify the following propositions:

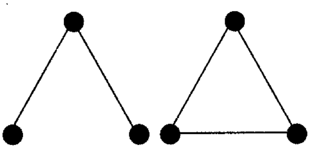
(i) In an orthomodular lattice it is not possible to find a chain with length greater than the maximum of the cardinality of cliques.

(ii) If there are 3 cliques such that $C_2 \subseteq C_1 \cup C_3$, then

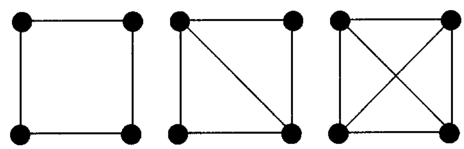
$$C_1 \cap C_3 = \emptyset \Rightarrow (C_1 - C_2) \cup (C_3 - C_2)$$

must be a clique under the assumption of the orthomodularity law.

By direct verification for the delimitation of the cardinality of cliques 2 and 3, respectively, we obtain the following forbidden subgraphs:



Case 2



Case 3

and these assumptions are sufficient.

Open problem: To find a counterexample for the hypothesis that the assumptions described above are sufficient or to prove the hypothesis.

REFERENCES

Birkhoff, G. (1967). *Lattice Theory*, AMS, Providence, Rhode Island.

McLaren, M. D. (1964). Atomic orthocomplemented lattices, *Pacific Journal of Mathematics*, **14**, 597.

Zapatrin, R. R. (1990). Graphs, polarities and finite ortholattices, *Second Winter School on Measure Theory Liptovsky Jan*, pp. 213–218.

Zapatrin, R. R. (n.d.). Boolean machinery for quantum logics, *Foundations of Physics*, submitted.